

The Briançon-Skoda Theorem for pseudo-rational and Du Bois singularities

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Def | R noetherian, $I \in R$, $Y \rightarrow \operatorname{Spec} R$ normalized blowup of I , $I\mathcal{O}_Y = \mathcal{O}_Y(-E)$

Then the integral closure of I in R is

$$\bar{I} := \Gamma(Y, \mathcal{O}_Y(-E)) \cap R$$

$\uparrow$ when R not normal

Rule

(1) if $Y \xrightarrow{\pi} X$ proper, birational X, Y normal varieties $D \geq 0$ Cartier on Y then

$\pi_* \mathcal{O}_Y(-D)$ is a sheaf of integrally closed ideals on X

Ex | $\mathcal{J}(X)$ multiplier ideal is integrally closed

$$* I \subseteq \bar{I}$$

(2) R domain, $R \subseteq V \subseteq \operatorname{Frac}(R)$ valuation ring

$$\bar{I} = \bigcap_V \ker(R \rightarrow R/I \otimes V)$$

Theorem (Briançon-Skoda '74): $I = (f_1, \dots, f_n) \in \mathbb{C}[[x_1, \dots, x_d]]$. Then $\forall k \geq 1$

$$\overline{I^{n+k-1}} \subseteq I^k$$

Proof uses a result of Skoda in L^2 analysis - affnor to algebraists

Theorem (Lipman-Sarason '81): $I = (f_1, \dots, f_n) \in R$ local, dim d , regular.

Then $\forall k \geq 1$

$$\overline{I^{n+k-1}} \subseteq I^k$$

Aside: Skoda's Theorem in modern algebraic language says that $\forall k \geq 1$

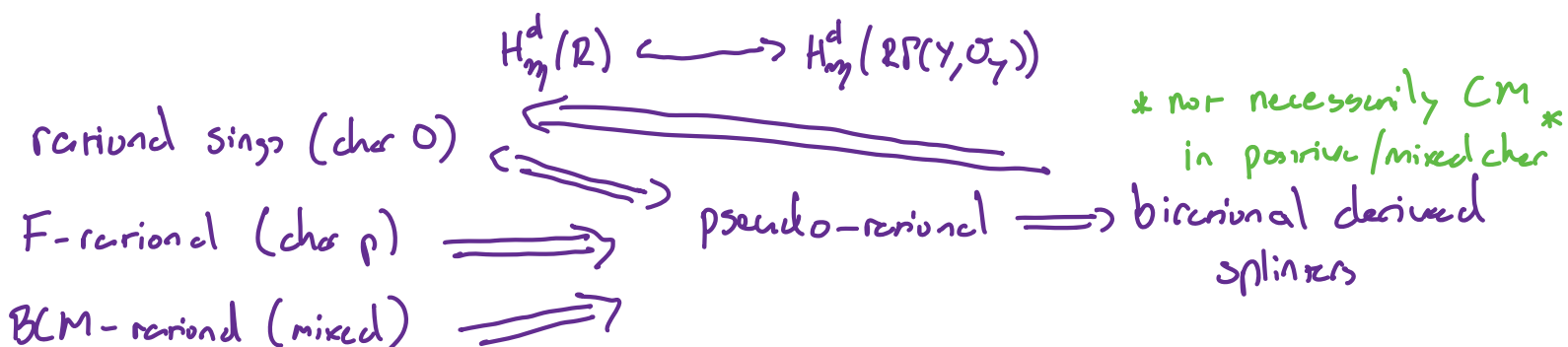
$$\overline{I^{n+k-1}} \subseteq \mathcal{J}(I^{n+k-1}) \subseteq I^k \mathcal{J}(I^{n-1}) \subseteq I^k$$

• Lipman '93: phrased Skoda's theorem algebraically and showed it follows from a Grauert-Riemenschneider type vanishing - Cartwright uses Kodaira vanishing to settle this for local rings etc / $\mathbb{C}$

- Ein-Lazarsfeld '99: give a global algebraic proof of Kodaira's theorem using Kawamata-Viehweg vanishing - use to get a geometric effective nullstellensatz

Moving on to singular rings

Def $(R, \mathfrak{m})$ local ring has pseudo-rational singularities if R is normal, Cohen-Macaulay, analytically unramified ($= \hat{R}$ reduced) and $\forall \pi: Y \rightarrow \text{Spec } R$ proper birational w/ Y normal, integral have



Theorem (Lipman - Teissier '81): $(R, \mathfrak{m})$ pseudo-regular of dim d , $I \subseteq R$. $\forall k \geq 1$

$$\overline{I^{d+k-1}} \subseteq I^k$$

Remark: d is suboptimal! Should be able to replace w/ $n \leq d$

Theorem (Aberbach - Huneke '01): if R excellent F -regular $\overset{\text{char } p > 0}{\wedge}$, $I = (f_1, \dots, f_n)$. $\forall k \geq 1$

$$\overline{I^{n+k-1}} \subseteq I^k$$

$\Rightarrow$ using reduction mod p , settles BS in char 0 (for $R \supseteq \mathbb{Q}$ w/ regular sing.)

$\exists$ examples of pseudo-regular but not F -regular

Def R reduced is a birationally derived splinter if $\forall \pi: Y \rightarrow \text{Spec } R$

proper birational the map $R \rightarrow R\Gamma(Y, \mathcal{O}_Y)$ splits in $D^b(R)$

$\nwarrow \quad \nearrow$
composition = id_R

Thm (Ma - M - R.G. - Schwede '26): R a birational derived splitter, $I = (f_1, \dots, f_n)$

Then $\forall k \geq 1$

$$\overline{I^{n+k-1}} \subseteq I^k$$

Proof: Focus on $k=1$. Take $Y \rightarrow \text{Spec } R$ normalized blowup of I , $I\mathcal{O}_Y = \mathcal{O}_Y(-E)$

Consider Koszul cx of $f_1, \dots, f_n$ on $\mathcal{O}_Y$

$$K_*(\underline{f}; \mathcal{O}_Y): 0 \rightarrow \mathcal{O}_Y \xrightarrow{\quad} \mathcal{O}_Y^{\oplus n} \rightarrow \dots \rightarrow \mathcal{O}_Y^{\oplus n} \xrightarrow{\quad} \mathcal{O}_Y \rightarrow 0$$

n B

Inside of this, there sits an exact subcomplex

$$B: 0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Y^{\oplus}(-E) \rightarrow \dots \rightarrow \mathcal{O}_Y^{\oplus n}(-(n-1)E) \rightarrow \mathcal{O}_Y(-n) \rightarrow 0$$

Then we have a map

$$\mathcal{O}_Y(-nE) \rightarrow B \rightarrow K_*(\underline{f}; \mathcal{O}_Y)$$

This is 0 in $D^b(Y)$. Apply $R\Gamma(Y, -)$ to get

$$R\Gamma(Y, \mathcal{O}_Y(-nE)) \xrightarrow{0} K_*(\underline{f}; R\Gamma(Y, \mathcal{O}_Y))$$

Taking $H_0(-)$ $I^{KH} = \ker(H_0(\underline{f}; R) \rightarrow H_0(\underline{f}; R\Gamma(Y, \mathcal{O}_Y)))$ $\parallel$ I is Koszul-Hirsch

$$\overline{I^n} = R \cap \Gamma(Y, \mathcal{O}_Y(-nE)) \xrightarrow{0} H_0(\underline{f}; R\Gamma(Y, \mathcal{O}_Y)) \text{ "closed"}$$

$$\begin{array}{ccc} 0 & \searrow & \\ & R/\underline{f} = H_0(\underline{f}; R) & \nearrow \end{array}$$

split injection b/c R is birational derived splitter

Therefore $\overline{I^n} \subseteq I$. For $k > 1$, replace $K_*(\underline{f}; -)$ w/ $L^k(\underline{f}; -)$
the Buchsbaum-Eisenbud cx / Eagon-Northcott cx to get $\overline{I^{n+k-1}} \subseteq I^k$

Q | What about Du Bois? Rational sing. $\Rightarrow$ Du Bois

The key of our proof is that

$$\mathcal{O}_Y(-nE) \xrightarrow{0} K_*(\underline{f}; \mathcal{O}_Y)$$

Twist by $\mathcal{O}_Y(-E)$

Then

$$\mathcal{O}_Y(-(n+1)E) \xrightarrow{0} K_*(\mathbb{A}; \mathcal{O}_Y(-E))$$

Take $R\Gamma(Y, -)$

$$R\Gamma(Y, \mathcal{O}_Y(-(n+1)E)) \xrightarrow{0} K_*(\mathbb{A}; R\Gamma(Y, \mathcal{O}_Y(-E)))$$

Consider $F = \text{reduced preimage of } V(\sqrt{I}) \text{ in } Y$. Hence Δ in $D^b(R)$

$$\begin{array}{ccc} F & \longrightarrow & Y \\ \downarrow & & \downarrow \\ V(\sqrt{I}) & \longrightarrow & \text{Spec } R \end{array} \quad \begin{array}{ccc} & \swarrow R & \\ D_I^\bullet & \xrightarrow{\quad} & R\Gamma(Y, \mathcal{O}_Y) \oplus R/\sqrt{I} \longrightarrow R\Gamma(F, \mathcal{O}_F) \\ \uparrow & & \\ R\Gamma(Y, \mathcal{O}_Y(-E)) & \text{from } \mathcal{O}_Y(-E) \subseteq \mathcal{I}_F \subseteq \mathcal{O}_Y & \end{array}$$

Then

$$R\Gamma(Y, \mathcal{O}_Y(-(n+1)E)) \xrightarrow{0} K_*(\mathbb{A}; D_I^\bullet)$$

$K_*(\mathbb{A}; R)$ ↗ coming from $R \rightarrow D_I^\bullet$
when R Du Bois, split injection

Key: $R \xrightarrow{\sim} \Omega_R^\bullet \xrightarrow{\quad} D_I^\bullet$

Taking 0^{th} cohomology, get $\overline{I}^{n+1} \subseteq I$

$$\overline{I}^{n+k} \subseteq I^k$$

Def R reduced blowing square splines if $R \rightarrow D_I^\bullet$ split $\forall I$